NUMERICAL SOLUTION OF A PROBLEM INVOLVING THE INTERACTION OF A SHOCK WAVE WITH A CYLINDER IN A SUPERSONIC FLOW

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We describe a difference scheme of second-order accuracy for calculating axially symmetric flows containing shock waves, and we apply our computational results to the interaction of a shock wave with the end of a cylinder moving at supersonic speed.

Because of the particular features of nonstationary gas flows, in which there exists the possibility of a simultaneous development and unbounded growth of jumps, it is necessary to apply continuous computational methods in the solution of engineering problems. At the present time several such methods are known, differing from one another in the formulational scheme employed and the accuracy of the results which they ensure [1-5]. In this connection, it is customary to use methods of first-order accuracy [1-4] in calculating two-dimensional axially-symmetric flows. In this paper we describe a two-dimensional difference scheme of two-step type, which is of second-order accuracy. The scheme we present is applied to the solution of a "wave-on-a-wave" type of problem, namely, the calculation of shock wave processes which arise when a nonstationary shock wave impinges on the stationary wave surrounding the frontal portion of a cylinder moving with supersonic speed.

In writing difference equations to allow continuous calculations of the shock waves, we first write the system of two-dimensional flow equations in divergent form:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\nu}{y}\rho v=0,$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+p)+\frac{\partial}{\partial y}(\rho uv)+\frac{\nu}{y}\rho uv=0,$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho v^2+p)+\frac{\nu}{y}\rho v^2=0,$$

$$\frac{\partial e}{\partial t}+\frac{\partial}{\partial x}(e+p)u+\frac{\partial}{\partial y}(e+p)v+\frac{\nu}{y}(e+p)v=0, \tag{1}$$

where $\nu = 1$ in the axially symmetric case and $\nu = 0$ in the planar case. In vector notation we may write the system (1) in the form

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}F(f)+\frac{\partial}{\partial y}G(f)+\frac{\nu}{y}H(f)=0. \tag{2}$$

Two-Dimensional Difference Scheme of Second-Order Accuracy

For a difference approximation of the system (2) we employ 9 nodes of a supporting layer and one node of a computing layer. The calculations are carried out in two stages.

In a preliminary halfstep we implement a four-point scheme of first-order accuracy, described by the vector equation

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$$f^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l+\frac{1}{2}} = \frac{1}{4}\left(f^n_{k,l} + f^n_{k+1,l} + f^n_{k+1,l+1} + f^n_{k,l+1}\right) - \frac{\Delta t}{4\Delta x}\left(F^n_{k+1,l+1} + F^n_{k+1,l} - F^n_{k,l+1} - F^n_{k,l}\right) \tag{3}$$

$$- \frac{\Delta t}{4\Delta y}\left(G^n_{k+1,l+1} + G^n_{k,l+1} - G^n_{k,l} - G^n_{k+1,l}\right) - \frac{\nu\Delta t}{8y_{l+\frac{1}{2}}}\left(H^n_{k,l} + H^n_{k+1,l} + H^n_{k+1,l+1} + H^n_{k,l+1}\right).$$

The values of the vector function f calculated in this preliminary halfstep are then used in a second final halfstep. The difference equation for the second stage describes a spatial variant of the "criss-cross" scheme with artificial viscosity:

$$f^{n+1}_{k,l} = f^n_{k,l} - \frac{\Delta t}{2\Delta x}\Big(F^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l+\frac{1}{2}} + F^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l-\frac{1}{2}}$$

$$- F^{n+\frac{1}{2}}_{k-\frac{1}{2},\, l+\frac{1}{2}} - F^{n+\frac{1}{2}}_{k-\frac{1}{2},\, l-\frac{1}{2}}\Big) - \frac{\Delta t}{2\Delta y}\Big(G^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l+\frac{1}{2}} + G^{n+\frac{1}{2}}_{k-\frac{1}{2},\, l+\frac{1}{2}}$$

$$- G^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l-\frac{1}{2}} - G^{n+\frac{1}{2}}_{k-\frac{1}{2},\, l-\frac{1}{2}}\Big) + Q^n_{k+\frac{1}{2},\, l} - Q^n_{k-\frac{1}{2},\, l} + R^n_{k,\, l+\frac{1}{2}} - R^n_{k,\, l-\frac{1}{2}}$$

$$- \frac{\nu\Delta t}{4y_l}\Big(H^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l+\frac{1}{2}} + H^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l-\frac{1}{2}} + H^{n+\frac{1}{2}}_{k-\frac{1}{2},\, l+\frac{1}{2}} + H^{n+\frac{1}{2}}_{k-\frac{1}{2},\, l-\frac{1}{2}}\Big). \tag{4}$$

In Eq. (4), Q and R define the vector artificial viscosity in the x and y directions, respectively.

The type of artificial viscosity to be chosen is decided with the help of numerical experimentation. The viscosity selected must smooth out the oscillations of the solution in a neighborhood of the shock wave front, and also the sum of the supplementary terms in Eq. (4) must not be less than the third order of smallness. It has been found that the artificial viscosity defined by the following relations possesses good stabilizing properties:

$$Q^n_{k+\frac{1}{2},\, l} = \frac{\omega}{2}\left|\varkappa^n_{k+1,l} - \varkappa^n_{k,l}\right|\left(f^n_{k+1,l} - f^n_{k,l}\right),$$

$$R^n_{k,\, l+\frac{1}{2}} = \frac{\omega}{2}\left|\lambda^n_{k,l+1} - \lambda^n_{k,l}\right|\left(f^n_{k,l+1} - f^n_{k,l}\right), \tag{5}$$

$$\varkappa = \frac{\Delta t}{\Delta x}\left(|u| + \bar{a}\right), \qquad \lambda = \frac{\Delta t}{\Delta y}\left(|v| + \bar{a}\right).$$

In our computations we used values of the viscosity coefficient in the range $0.5 \leq \omega \leq 2.0$.

Investigation of the Stability of the Difference Equations

We use Fourier's method to establish bounds on the time step Δt which ensure stability of the solution with respect to small perturbations. To apply Fourier's method we linearize the system (2), obtaining a linear system with constant coefficients:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{6}$$

where $A = dF/df$, $B = dG/df$ are Jacobi matrices of the vectors F and G, considered at an arbitrary point of the flow region. Since the system (2) is of hyperbolic type, we may consider the matrices A and B to be symmetric. The difference scheme (3)-(4), when applied to the linear system (6), may be written in the form of the explicit relation

$$U^{n+1}_{k,l} = U^n_{k,l} - \frac{\Delta t}{8\Delta x}A\left(U^n_{k+1,\, l-1} + 2U^n_{k+1,l} + U^n_{k+1,l+1} - U^n_{k-1,l-1} - 2U^n_{k-1,l} - U^n_{k-1,l+1}\right)$$

$$- \frac{\Delta t}{8\Delta y}B\left(U^n_{k-1,l+1} + 2U^n_{k,l+1} + U^n_{k+1,l+1} - U^n_{k-1,l-1} - 2U^n_{k,l-1} - U^n_{k+1,l-1}\right)$$

$$+ \frac{\Delta t^2}{8\Delta x^2}A^2\left(2U^n_{k+1,l} - 4U^n_{k,l} + 2U^n_{k-1,l} + U^n_{k+1,l+1} - 2U^n_{k,l+1} + U^n_{k-1,l+1} + U^n_{k+1,l-1} - 2U^n_{k,l-1} + U^n_{k-1,l-1}\right)$$

$$+\frac{\Delta t^2}{8\Delta x\Delta y}(AB+BA)\left(U^n_{k+1,l+1}-U^n_{k+1,l-1}-U^n_{k-1,l+1}+U^n_{k-1,l-1}\right)$$

$$+\frac{\Delta t^2}{8\Delta y^2}B^2\left(2U^n_{k,l+1}-4U^n_{k,l}+2U^n_{k,l-1}+U^n_{k+1,l+1}-2U^n_{k+1,l}+U^n_{k+1,l-1}+U^n_{k-1,l-1}-2U^n_{k-1,l}+U^n_{k-1,l+1}\right). \quad (7)$$

We may study how the linearized system (7) reacts to a vector-function of the form

$$U^n_{k,l}=\xi^n\exp\left[i(k\varphi+l\psi)\right], \quad (8)$$

which describes simple harmonic oscillations. If the function (8) is assigned at the n-th layer, a periodic function is obtained at the (n + 1)-th layer the vector amplitudes of which may be expressed by a relationship which results from substituting the function (8) into the difference equation (7):

$$\xi^{n+1}=S\xi^n,$$

where S is the transfer matrix of the difference scheme, defined by the expression

$$S=E-2ai\cos^2\frac{\psi}{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}-2bi\cos^2\frac{\varphi}{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2}$$

$$-2a^2\sin^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}-2(ab+ba)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2}-2b^2\sin^2\frac{\psi}{2}\cos^2\frac{\varphi}{2}, \quad (9)$$

where E is the unit matrix; $a = (\Delta t/\Delta x)A$; $b = (\Delta t/\Delta y)B$.

The scheme is stable if the norm of the transfer matrix does not exceed unity. From the definition of the matrix norm this condition is valid if

$$|(Sq,\ q)|^2\leqslant 1 \quad (10)$$

for all unit vectors q.

For convenience in the work which follows we write the transfer matrix in the form

$$S=C+iJ, \quad (11)$$

where

$$C=E-2D^2;\quad J=2\cos\frac{\varphi}{2}\cos\frac{\psi}{2}D;$$

$$D=a\sin\frac{\varphi}{2}\cos\frac{\psi}{2}+b\sin\frac{\psi}{2}\cos\frac{\varphi}{2}.$$

In addition, we let $|Dq|^2 = \delta$, $|aq|^2 = \alpha$, $|bq|^2 = \beta$. The following relations are valid:

$$c^2=(Cq,\ q)^2=(1-2\delta)^2,$$

$$j^2=(Jq,\ q)^2=4\cos^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}(Dq,\ q)^2\leqslant 4\cos^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}\delta, \quad (12)$$

where the last estimate holds by virtue of Schwarz's inequality. From the relations (12) we immediately obtain an estimate for the left side of inequality (10):

$$|(Sq,\ q)|^2=c^2+j^2\leqslant 1-4\delta\left(1-\delta-\cos^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}\right). \quad (13)$$

We now need to explain the condition of nonnegativity of the expression appearing in parentheses in the right member of inequality (13).

The quantity δ may be estimated as follows:

$$\delta=\left|\left(a\sin\frac{\varphi}{2}\cos\frac{\psi}{2}+b\sin\frac{\psi}{2}\cos\frac{\varphi}{2}\right)q\right|^2\leqslant\left(|aq|\sin\frac{\varphi}{2}\cos\frac{\psi}{2}+|bq|\sin\frac{\psi}{2}\cos\frac{\varphi}{2}\right)^2$$

$$\leqslant\max(\alpha,\ \beta)\left(\sin\frac{\varphi}{2}\cos\frac{\psi}{2}+\sin\frac{\psi}{2}\cos\frac{\varphi}{2}\right)^2. \quad (14)$$

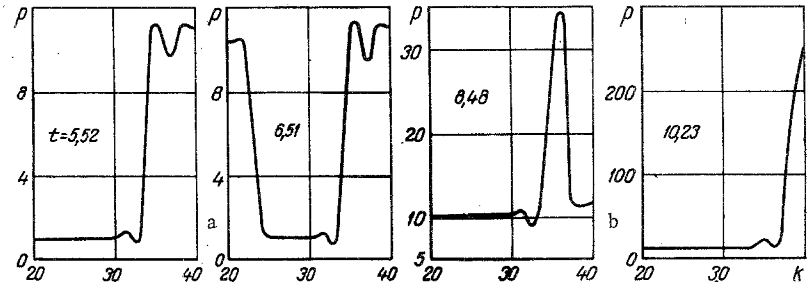

Fig. 1. Axial pressure profiles: a) before and after the appearance of the incident shock wave; b) during the interaction of the incident wave with the bow wave, and at the instant of impact with the end of the cylinder.

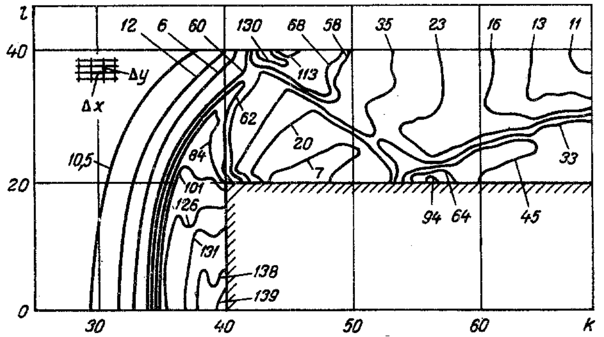

Fig. 2. Isobars for the second stationary mode: $t = 18.15$; $\Delta x = 1.0$; $\Delta y = 0.5$.

By virtue of the trigonometric relation

$$\left(\sin\frac{\varphi}{2}\cos\frac{\psi}{2}+\sin\frac{\psi}{2}\cos\frac{\varphi}{2}\right)^2 \leqslant 2\left(1-\cos^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}\right)$$

we may extend the estimate (14):

$$\delta \leqslant \max(\alpha,\beta)\quad 2\left(1-\cos^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}\right). \tag{15}$$

Using the inequality (15) we may write inequality (13) in the form

$$c^2+j^2 \leqslant 1-4\delta\left(1-2\max(\alpha,\beta)\right)\left(1-\cos^2\frac{\varphi}{2}\cos^2\frac{\psi}{2}\right). \tag{16}$$

The right side of inequality (16) does not exceed unity for $\alpha \le 1/2$, $\beta \le 1/2$, or for

$$\left(\frac{\Delta t}{\Delta x}A\right)^2 \leqslant \frac{1}{2}E,\quad \left(\frac{\Delta t}{\Delta y}B\right)^2 \leqslant \frac{1}{2}E. \tag{17}$$

The inequalities (17) express the conditions for stability of the difference scheme. In practice it is convenient to write them in the equivalent form

$$\varkappa_m \leqslant 1/\sqrt{2},\quad \lambda_m \leqslant 1/\sqrt{2}\quad (m=1,\ 2,\ 3,\ 4). \tag{18}$$

The stability conditions (17) are the best possible, i.e., they are not only sufficient but also necessary.

In the study of problems of the "wave-on-a-wave" type it is necessary to take into account the interaction of two shock waves, the incident wave and the bow wave which forms ahead of the body during its motion. As a result of the collision of the incident wave with the stationary bow wave in the subsonic region, two shock waves are formed, moving in the direction of the body; these waves are usually separated by a contact surface. During the action of the strong forward wave the pressure on the surface of the body increases sharply. This increase amounts to an impulse since the rarefaction waves from the periphery of the cylinder lead to a subsequent lowering of the pressure. After this transient process, a new stationary flow mode is established, corresponding to the flow parameters behind the incident wave front.

Actually, in making the calculations two subsidiary problems must be solved: the flow over the body in the initial supersonic flow must be determined; and the new supersonic flow mode over the body, resulting from impingement of the shock wave, must be determined. A simultaneous solution of these problems, by means of a continuous computational method, may be effected in the following way. As the initial field of values we take the parameters of the medium traversed by the first shock wave. After the collision of the wave with the body a retrograde motion of the reflected wave occurs and a stationary flow is formed. After the refracted wave is formed and the functions on the mesh no longer vary, a change in the flow parameters at the boundary (entrant) cells of the mesh is made, the aim being to obtain the second shock wave moving with respect to the stationary flow. The parameters of this second shock are obtained through use of the Hugoniot relations.

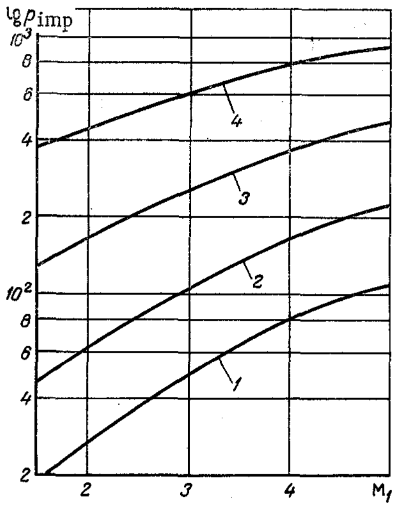

Fig. 3. Dependence of pressure impulse on M_1 and M_2: 1) $M_2 = 1.5$; 2) $M_2 = 2.0$; 3) $M_2 = 3.0$; 4) $M_2 = 5.0$.

When the incident shock wave meets the bow wave, the most important phase of the nonstationary interaction begins; at this time the values of the pressure on the body are a maximum. This phase terminates with the establishment of a second stationary mode.

Digital computer results of the numerical solution are shown in Figs. 1-3. In these particular calculations, $\Delta x = 0.5\Delta y = 0.05\,r$, where r is the cylinder radius. The entire region of the mesh contained 40×80 cells. Parameters of the incident flow were chosen so that $p_\infty = 1$, $\rho_\infty = 1$. As the scaling length we took the mesh step Δx. Thus the calculations were made in terms of the dimensionless variables pp_∞^{-1}, $\rho\rho_\infty^{-1}$, $up_\infty^{-1/2}\rho_\infty^{1/2}$, $vp_\infty^{-1/2}\rho_\infty^{1/2}$, $x\Delta x^{-1}$, $y\Delta x^{-1}$, $tp_\infty^{1/2}\rho_\infty^{-1/2} \cdot \Delta x^{-1}$. For the specific heat ratio we used $\gamma = 1.4$. The flow Mach number M_1 and the Mach number M_2 of the incident shock wave were varied.

Figure 1 shows axial pressure profiles for the case $M_1 = M_2 = 3$ at the following characteristic time instants: 1) for the first flow mode ($t = 5.52$); 2) at the instant of release of a shock wave into the stationary flow ($t = 6.51$); 3) during the interaction of the incident shock wave with the bow wave ($t = 8.48$); 4) during interaction of the incident shock wave with the end of the cylinder ($t = 10.23$).

Figure 2 presents isobars at an instant of time when the flow in the neighborhood of the cylinder end is close to being stationary ($t = 18.15$). In Fig. 2 the reflection of the bow wave from the channel walls and the region of rarefaction at the lateral wall of the cylinder can be seen.

A nomogram of pressure impulse values as a function of M_1 and M_2 is shown in Fig. 3.

NOTATION

x	is the space coordinate calculated along the flow axis;
y	is the space coordinate measured from the flow axis;
t	is the time;
p	is the pressure;
ρ	is the density;
u	is the axial component of the velocity vector;
v	is the radial component of velocity;
e	is the total energy per unit volume, $e = \rho(u^2 + v^2)/2 + p/(\gamma - 1)$;
l	is the index of the node in the y direction;
γ	is the heat capacity ratio;
a	is the velocity of sound;
Δx	is the step in x on the difference grid;
Δy	is the step in y on the grid;
Δt	is the time step;
k	is the index of the grid node for which values are calculated in the x direction;
n	is the time index of the grid.

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